647510 **MSC2MatC202x** **Seat No :________**

M.Sc.(Maths.) Semester - 2 (*CBCS*) Examination
April/May-2023 (NEW COURSE)
Complex Analysis (Core)

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1(A) Answer the following questions. (Any Two) (10)

1. Prove that $e^{z+w} = e^z .\ e^w$ for all $z, w \in \mathbb{C}$ and $\overline{e^z} = e^{\bar{z}}$ for all $z \in \mathbb{C}$.
2. Let G & Ω be two open subsets of $\mathbb{C}$ and $f: G \to \mathbb{C}$ & $g: \Omega \to \mathbb{C}$ be differentiable functions and $g'(w) \neq 0; \forall\, w \in \Omega$ with $f(G) \subset \Omega$ and $g\big(f(z)\big) = z, \forall\, z \in G$ then $f: G \to \mathbb{C}$ is differentiable and $f'(z) = \frac{1}{g'(f(z))}; \forall z \in G$.
3. Let $M = \{S: S\ is\ bilinear\ map\}$ then prove that M forms group under the binary operation of composition of mappings.

Q.1(B) Answer the following question. (Any One) (04)

1. Is $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \bar{z}, \forall z \in \mathbb{C}$ analytic? Why?
2. Find radius of convergence of exponential series.

Q.2(A) Answer the following questions. (Any Two) (10)

1. Let γ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma; a)$ is constant for a belonging to a component of $G = \mathbb{C} \backslash \{\gamma\}$. Also, $n(\gamma; a) = 0$ for a belonging to the unbounded component of G.
2. State and prove Cauchy's Integral formula 1st version.
3. If G is an open set in $\mathbb{C}$, $\gamma : [a, b] \to G$ is a rectifiable path, and $f : G \to \mathbb{C}$ is continuous then for every $\varepsilon > 0$ there is a polygonal path Γ in G such that $\Gamma(a) = \gamma(a),\ \Gamma(b) = \gamma(b)$, and $\left|\int_\gamma f - \int_\Gamma f\right| < \varepsilon$.

Q.2(B) Answer the following question. (Any One) (04)

1. Find $\int_\gamma f(z)dz$, where $\gamma : [0,2\pi] \to \mathbb{C}$ to be $\gamma(t) = e^{it}$ and

   $f(z) = \frac{1}{z}, z \neq 0.$

2. Define the following terms:

   (I) Index of closed curve. **(II)** Function of Bounded Variations.

Q.3(A) Answer the following questions. (Any Two) (10)

1. State and prove Morera's theorem.
2. Let $G \subseteq \mathbb{C}$ be a region and $f: G \to \mathbb{C}$ be an analytic function such that $|f(z)| \leq |f(a)|, \forall z \in G$ and for some $a \in G$ then $f: G \to \mathbb{C}$ is a constant map.
3. Let G be a connected open set and let $f : G \to \mathbb{C}$ be an analytic function such that the set $\{z \in G : f(z) = 0\}$ has a limit point in G, then prove that there is a point a in G such that $f^n(a) = 0$ for each $n \geq 0$.

Q.3(B) Answer the following question. (Any One) (04)

1. Does there exist an analytic function $f : D \to \mathbb{C}$ such that $|f(z)| = e^{|z|}; \forall z \in D$? Justify.
2. State and prove Liouville's theorem.

Q.4(A) Answer the following questions. (Any Two) (10)

1. If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then $\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$ is an integer.
2. State and prove counting zero principle.
3. Let $f : \mathrm{D} \to \mathrm{D}$ be analytic and one-one map of D onto D and suppose that $f(a) = 0$. Then there is a complex number c with $|c| = 1$ such that $f = c\phi_a$. Where $\phi_a(z) = \frac{1-a}{1-\bar{a}z}$

Q.4(B) Answer the following question. (Any One) (04)

1. Does there exist an analytic function $f : \mathrm{D} \to \mathrm{D}$ such that $f\left(\frac{1}{2}\right) = \frac{3}{4}$ and $f'\left(\frac{1}{2}\right) = \frac{2}{3}$?
2. Evaluate $\int_\gamma \frac{f'(z)}{f(z)} dz$, where $f(z) = (z^2+1)^3$ and $\gamma(t) = 2e^{it}, 0 \leq t \leq 2\pi$.

Q.5(A) Answer the following questions. (Any Two) (10)

1. State and prove Riemann's theorem on removable singularities.
2. Prove that an isolated singularity a of f is an essential singularity iff for any $\delta > 0$ the set $\{f(z): 0 < |z-a| < \delta\}$ is dense in $\mathbb{C}$.
3. State and prove Argument principle.

Q.5(B) Answer the following question. (Any One) (04)

1. State Taylor's theorem.
2. Find residue of function $f(z) = z + \frac{1}{z}$ at 0.

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